JOURNAL OF COMPUTATIONAL PHYSICS 21, 356–360 (1976)

Note

Gaussian Quadrature Formulas for the Numerical Calculation of Integrals with Logarithmic Singularity

The double integral

$$I = \int_0^1 \int_0^1 f(x, y)/(x - y)\, dx\, dy \qquad (1)$$

with Cauchy-type singularity arises in the study of aerodynamic or hydrodynamic load on a lifting body [1, 2]. Applying an ingenious substitution, Song [3] has transformed (1) into

$$I = I_1 + I_2 \qquad (2)$$

where

$$I_1 = \int_0^1 \int_0^1 [f(yt, y) - f(yt - t + 1, y)]/(t - 1)\, dt\, dy \qquad (3)$$

and

$$I_2 = \int_0^1 \ln[(1 - y)/y]\, f(y, y)\, dy. \qquad (4)$$

If f and $\partial f/\partial x$ are bounded for $-1 \leqslant x, y \leqslant 1$, the double integral (3) is a proper integral which can be computed using standard cubature rules [4]. The integral (4) has two logarithmic end point-singularities. Using the substitution $x = 1 - 2y$, it can be transformed into

$$I_2 = \int_{-1}^1 \phi(x) \ln\,[(1 + x)/(1 - x)]\, dx \qquad (5)$$

where

$$\phi(x) = f[(1 - x)/2, (1 - x)/2]/2.$$

It is the purpose of this note to present a table of abscissas and weights of gaussian quadrature formulas for the evaluation of (5).

The weight function

$$w(x) = \ln[(1 - x)/(1 + x)] \qquad (6)$$

in (5) is an odd function and consequently [5], the $2N$-point gaussian quadrature formula

$$\int_{-1}^{1} \phi(x) \ln[(1+x)/(1-x)]\, dx \simeq \sum_{k=1}^{N} w_k[\phi(x_k) - \phi(-x_k)], \qquad N = 1, 2,... \quad (7)$$

exists. The abscissas x_k and weights w_k are given by $x_k = u_k^{1/2}$ and $w_k = v_k/x_k$, $k = 1, 2,..., N$, where u_k and v_k are the abscissas and weights of the Gaussian N-point formula

$$\int_{0}^{1} \ln[(1+u^{1/2})/(1-u^{1/2})]\, \psi(u)\, du \simeq 2 \sum_{k=1}^{N} v_k \psi(u_k). \quad (8)$$

Since the weight function in (8) is nonnegative and since all moments

$$m_i = \int_{0}^{1} \ln[(1+u^{1/2})/(1-u^{1/2})]\, u^i\, du = 2(i+1)^{-1} \sum_{j=1}^{i+1} (2j-1)^{-1}, \quad (9)$$

$i = 0, 1, 2,...$ exist, the quadrature formula (8) has real abscissas u_k in the interval (0, 1), and has positive weights v_k. Several algorithms for the computation of u_k and v_k using the moments (9) are described in the literature [6, 7]. However, it is well known that the construction of gaussian formulas, using ordinary moments, is strongly ill-conditioned [8]. In order to have a well-conditioned problem, we need the modified moments

$$M_i = \int_{0}^{1} \ln[(1+u^{1/2})/(1-u^{1/2})]\, T_i(2u-1)\, du \quad (10)$$

where T_i is the Chebyshev polynomial of the first kind and of degree i. Substituting $u = x^2$ into (10) and using well-known properties of the Chebyshev polynomials [9], we obtain

$$M_i = \int_{-1}^{1} \ln[(1+x)/2][T_{2i+1}(x) + T_{2i-1}(x)]\, dx. \quad (11)$$

A recurrence relation for the computation of (11) is given in [10]. Now, the algorithm presented by Gautschi [8] can be applied for the construction of the Gaussian quadrature rule (8).

The abscissas and weights of the Gaussian quadrature formula (7), for $N = 2, 4,..., 18$ are presented in Table I. Tables for $N = 20, 30, 40$ and 50 are given in [11].

All computations are carried out on a IBM 370/158 computer, using extended precision.

TABLE I

Abscissas x_k and Weights w_k of the Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x) \ln[(1+x)/(1-x)]\, dx \simeq \sum_{k=1}^{M/2} w_k[f(x_k) - f(-x_k)]$$

M	x_k	w_k
2	0.81649 65809 27726 03273 $E+0$	0.12247 44871 39158 90491 $E+1$
4	0.56318 76691 22512 76872 $E+0$	0.62695 54799 54730 78728 $E+0$
	0.92597 12867 85678 72798 $E+0$	0.69862 46915 34780 84383 $E+0$
6	0.42041 10764 80722 72115 $E+0$	0.35227 03074 39083 73384 $E+0$
	0.76119 37707 26794 21975 $E+0$	0.55528 47905 05106 01731 $E+0$
	0.96001 51738 62141 50383 $E+0$	0.44709 95344 77937 84775 $E+0$
8	0.33365 35398 87395 67960 $E+0$	0.22233 21078 25118 63411 $E+0$
	0.62837 67488 06064 89938 $E+0$	0.38899 24551 36776 39778 $E+0$
	0.85021 29500 95481 22025 $E+0$	0.44310 14540 61239 04719 $E+0$
	0.97492 63218 33825 69659 $E+0$	0.31248 89436 79900 60742 $E+0$
10	0.27606 36999 94800 19237 $E+0$	0.15231 38114 13026 26769 $E+0$
	0.53034 78820 33572 38501 $E+0$	0.27937 41795 62341 57830 $E+0$
	0.74291 51506 66348 11680 $E+0$	0.35575 14413 58796 91893 $E+0$
	0.89741 55601 15883 72284 $E+0$	0.35373 74928 24015 70174 $E+0$
	0.98278 34489 96595 67665 $E+0$	0.23203 84165 55131 98034 $E+0$
12	0.23523 12061 37427 81565 $E+0$	0.11062 28658 52286 28071 $E+0$
	0.45709 12029 94083 98090 $E+0$	0.20819 16986 83831 99093 $E+0$
	0.65301 34212 62185 31228 $E+0$	0.27962 08746 09215 00553 $E+0$
	0.81200 56852 61707 39870 $E+0$	0.31132 15150 28267 40435 $E+0$
	0.92536 73925 59699 90766 $E+0$	0.28719 27414 80154 08687 $E+0$
	0.98743 44128 00853 89022 $E+0$	0.17992 65156 78923 86866 $E+0$

Table continued

TABLE I (*continued*)

M	x_k					w_k				
14	0.20483	28534	68534	75454	$E+0$	0.83892	13645	14291	27332	$E-1$
	0.40088	24292	57998	36851	$E+0$	0.16037	48165	78229	39720	$E+0$
	0.57976	07843	62754	14763	$E+0$	0.22203	29007	65799	60517	$E+0$
	0.73385	68624	21490	26304	$E+0$	0.26130	99860	27607	25323	$E+0$
	0.85669	39320	52939	55269	$E+0$	0.26998	80777	65830	49807	$E+0$
	0.94326	51189	36524	17682	$E+0$	0.23748	76092	42455	35127	$E+0$
	0.99041	73891	29697	57843	$E+0$	0.14410	45651	23003	04415	$E+0$
16	0.18134	74152	23869	45608	$E+0$	0.65763	00894	67130	27291	$E-1$
	0.35662	16317	04242	97338	$E+0$	0.12701	41957	87818	31877	$E+0$
	0.51996	16184	25741	00152	$E+0$	0.17924	12113	73286	79103	$E+0$
	0.66592	46807	99086	80275	$E+0$	0.21788	50063	52746	30233	$E+0$
	0.78968	15473	91943	75103	$E+0$	0.23817	11296	28565	16212	$E+0$
	0.88719	80115	99990	89296	$E+0$	0.23463	14570	95793	57589	$E+0$
	0.95540	96880	96725	95703	$E+0$	0.19969	95165	72563	29585	$E+0$
	0.99244	62781	23931	67033	$E+0$	0.11833	96289	83263	84369	$E+0$
18	0.16266	90768	36973	61482	$E+0$	0.52916	46969	11446	00491	$E-1$
	0.32096	65862	75514	34488	$E+0$	0.10293	09352	84617	91751	$E+0$
	0.47064	29342	57102	99348	$E+0$	0.14714	18102	05732	21141	$E+0$
	0.60768	96353	12153	51892	$E+0$	0.18263	01095	91573	47410	$E+0$
	0.72845	30641	24958	38223	$E+0$	0.20639	57464	14112	21728	$E+0$
	0.82974	10530	32083	54617	$E+0$	0.21519	10357	55009	94695	$E+0$
	0.90892	25613	50708	79073	$E+0$	0.20510	35892	82450	35527	$E+0$
	0.96402	72358	73794	05301	$E+0$	0.17039	31796	97139	97576	$E+0$
	0.99388	96036	39699	92810	$E+0$	0.99137	43054	21713	03975	$E-1$

Acknowledgment

This work was supported by the NFWO (Belgium) under grant No. 2.0021.75.

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Received: January 5, 1976

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